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On the normalized Shannon capacity of a union

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Let $G_1 \times G_2$ denote the strong product of graphs G_1 and G_2, i.e. the graph on $V(G_1) \times V(G_2)$ in which (u_1, u_2) and (v_1, v_2) are adjacent if for each $i = 1, 2$ we have $u_i = v_i$ or $u_i v_i \in E(G_i)$. The Shannon capacity of G is $c(G) = \lim_{n\to\infty} \alpha(G^n)^{1/n}$, where G^n denotes the n-fold strong power of G, and $\alpha(H)$ denotes the independence number of a graph H. The normalized Shannon capacity of G is $C(G) = \frac{\log c(G)}{\log |V(G)|}$. Alon [1] asked whether for every $\epsilon > 0$ there are graphs G and G' satisfying $C(G), C(G') < \epsilon$ but with $C(G + G') > 1 - \epsilon$. We show that the answer is no.

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Despite much impressive work (e.g. [1], [3], [4], [5], [7]) since the introduction of the Shannon capacity in [8], many natural questions regarding this parameter remain widely open (see [2], [6] for surveys). Let $G_1 + G_2$ denote the disjoint union of the graphs G_1 and G_2. It is easy to see that $c(G_1 + G_2) \geq c(G_1) + c(G_2)$. Shannon [8] conjectured that $c(G_1 + G_2) = c(G_1) + c(G_2)$, but this was disproved in a strong form by Alon [1] who showed that there are n-vertex graphs G_1, G_2 with $c(G_i) < e^{c\sqrt{\log n \log\log n}}$ but $c(G_1 + G_2) \geq \sqrt{n}$. In terms of the normalized Shannon capacity, this implies that for any $\epsilon > 0$, there exist graphs G_1, G_2 with $C(G_i) < \epsilon$ but $C(G_1 + G_2) > 1/2 - \epsilon$. Alon [1] asked whether '1/2' can be changed to '1' here. In this short note we will give a negative answer to this question. In fact, the following result implies that '1/2' is tight.

Theorem 0.1. *If $C(G_1) \leq \epsilon$ and $C(G_2) \leq \epsilon$ then*

$$C(G_1 + G_2) \leq \frac{1+\epsilon}{2} + \frac{1-\epsilon}{2\log_2(|V(G_1)| + |V(G_2)|)}.$$

Proof. Let $N_i = |V(G_i)|$ for $i = 1, 2$. Fix a maximum size independent set I in $(G_1+G_2)^n$ for some $n \in \mathbb{N}$. We write $|I| = \sum_{S\subset[n]} |I_S|$, where $I_S = \{x = (x_1, \ldots, x_n) \in I : x_i \in V(G_1) \Leftrightarrow i \in S\}$.

To bound $|I_S|$, we may suppose that $S = [m]$ for some $0 \leq m \leq n$. Then I_S is an

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independent set in $G_1^m \times G_2^{n-m}$. As $C(G_1) \leq \epsilon$, by supermultiplicativity $\alpha(G_1^m) \leq N_1^{\epsilon m}$; similarly, $\alpha(G_2^{n-m}) \leq N_2^{\epsilon(n-m)}$. For any $x \in V(G_1)^m$, the set of $y \in V(G_2)^{n-m}$ such that $(x,y) \in I_S$ is independent in G_2^{n-m}, so $|I_S| \leq N_1^m N_2^{\epsilon(n-m)}$. Similarly, $|I_S| \leq N_1^{\epsilon m} N_2^{n-m}$.

We multiply these bounds: $|I_S|^2 \leq (N_1^m N_2^{n-m})^{1+\epsilon}$. Writing $\gamma = \frac{N_1}{N_1+N_2}$, we have

$$\begin{aligned}
\alpha((G_1+G_2)^n) = |I| &= \sum_{S \subset [n]} |I_S| \leq \sum_{m=0}^{n} \binom{n}{m} \left(N_1^{(1+\epsilon)/2}\right)^m \left(N_2^{(1+\epsilon)/2}\right)^{n-m} \\
&= (N_1^{(1+\epsilon)/2} + N_2^{(1+\epsilon)/2})^n \\
&= (\gamma^{(1+\epsilon)/2} + (1-\gamma)^{(1+\epsilon)/2})^n (N_1+N_2)^{(1+\epsilon)n/2} \\
&\leq 2^{(1-\epsilon)n/2} (N_1+N_2)^{(1+\epsilon)n/2},
\end{aligned}$$

as $\gamma^b + (1-\gamma)^b$ is maximized at $\gamma = 1/2$ for $0 < b < 1$ and $0 \leq \gamma \leq 1$. Therefore

$$C(G_1+G_2) = \lim_{n\to\infty} \frac{\log \alpha((G_1+G_2)^n)}{n \log (N_1+N_2)} \leq \frac{1+\epsilon}{2} + \frac{1-\epsilon}{2\log_2(N_1+N_2)}.$$

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